Renormalized-Generalized Solutions for the KPZ Equation

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Abstract

This work introduces a new notion of solution for the KPZ equation, in particular, our approach encompasses the Cole-Hopf solution. This new theory provides a pathwise notion of solution as well as a structured approximation theory. The developments are based on regularization arguments from the theory of distributions.

Key words: KPZ equation, Generalized functions, Generalized stochastic processes, Colombeau algebras, Stochastic partial differential equations.

MSC2000 subject classification: 60H15, 46F99.

1   Introduction

Models for interface growth have attracted much attention during the last two decades, in particular, these models have widespread application in many systems (see [5] and [24]), such as film growth by vapor or chemical deposition, bacterial growth, evolution of forest fire fronts, etc. For such systems,

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a major effort has been the identification of the scaling regimes and their classification into universality classes. Phenomenological equations, selected according to symmetry principles and conservation laws, are often able to reproduce various experimental data. Among these phenomenological equations, the one introduced by M. Kardar, G. Parisi and Y. Zhang ([21]) has been successful in describing properties of rough interfaces. This equation, the KPZ equation, is also related to Burgers' equation of turbulence and to directed polymers in random media (see [6]). The KPZ equation describes the evolution of the profile of the interface, $h(x, t)$, at position x and time t:

$$\begin{cases} \partial_t h(t, x) = \triangle h(t, x) + (\partial_x h(t, x))^2 + W(t, x) \\ h(0, x) = f(x) \end{cases} \tag{1}$$

where $W(t, x)$ is a space-time white noise. We refer to [14] and [26] for a more detailed historical account of the KPZ equation.

From a mathematical point of view, equation (1) is ill-posed since the solutions are expected to look locally like a Brownian motion. The term $(\nabla h(t, x))^2$ cannot make sense in a classical way. The theory of distributions developed by L. Schwartz in the 1950s has certain insufficiencies, in particular, the absence of a well defined multiplication of distributions (see for instance [25] and [29]).

L. Bertini and G. Giacomin in [6], proposed that the correct solutions for the KPZ equation are obtained by taking the logarithm of a solution of the stochastic heat equation with multiplicative noise. This is known as the Cole-Hopf solution for the KPZ equation.

The stochastic heat equation with multiplicative noise is the following Itô equation,

$$\begin{cases} dZ = \triangle Z \ dt + Z dW \\ Z(0, x) = e^{f(x)}. \end{cases} \tag{2}$$

It is well possed and its solution is always positive. The evidence for the Cole-Hopf solution is overwhelming even though it satisfies, at a purely formal

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level, the equation

$$\begin{cases} \partial_t u(t,x) = \ \triangle h(t,x) + \ (\partial_x h(t,x))^2 + W(t,x) - \infty \\ u(0,x) = f(x) \end{cases} \qquad (3)$$

where ∞ denotes an infinite constant.

The above remark indicates that a key problem for the theory of stochastic partial differential equations is to find an appropriate definition of solution for the KPZ equation which, at the same time, incorporates the Cole-Hopf solution.

The present article constructs spaces of generalized stochastic processes that can be used to renormalize the divergent term appearing in (1). Moreover, we prove that the Cole-Hopf solution becomes then a well defined solution to the KPZ equation. At the level of sequences, our space of generalized stochastic processes looks like a Colombeau space (see J. F. Colombeau [13]) and at the level of Schwartz distributions it looks like a generalized stochastic process space in the sense of Itô-Gelfand-Vilenkin (see [16] and [20]). A key ingredient in our study is the use of regularization techniques to define nonlinear operations in the theory of distributions, we refer the reader to [3], [25] and [31] for the background material.

The article is organized as follows: Section 2 reviews some basic facts on the standard cylindrical Wiener process. Section 3 introduces the spaces of renormalized semimartingales and renormalized distributions. Moreover, we give a notion of random generalized functions. Section 4 introduces the new concept of solution for the KPZ equation and proves existence of the solution which coincides with the Cole-Hopf solution.

2   Cylindrical Brownian motion

Let $\{W(t,\cdot) : t \in [0,T]\}$ be a standard cylindrical Wiener process in $L^2(\mathbb{R})$; it is canonically realized as a family of continuous processes satisfying:

1. For any $\varphi \in L^2(\mathbb{R})$, $\{W_t(\varphi), t \in [0, T]\}$ is a Brownian motion with variance $t \int \varphi^2(x) \ dx$,

2. For any $\varphi_1, \varphi_2 \in L^2(\mathbb{R})$ and $s, t \in [0, T]$,

$$\mathbb{E}(W_s(\varphi_1)(W_t(\varphi_2)) = (s \wedge t) \int \varphi_1(x)\varphi_2(x) \ dx.$$

Let $\{\mathcal{F}_t : t \in [0, T]\}$ be the σ-field generated by the P-null sets and the random variables $W_s(\varphi)$, where $\varphi \in \mathcal{D}(\mathbb{R})$ and $s \in [0, t]$. The predictable σ-field is the σ-field in $[0, T] \times \Omega$ generated by the sets $(s, t] \times A$ where $A \in \mathcal{F}_s$ and $0 \le s < t \le T$.

Let $\{v_j : j \in \mathbb{N}\}$ be a complete orthonormal basis of $L^2(\mathbb{R})$. For any predictable process $g \in L^2(\Omega \times [0, T], L^2(\mathbb{R}))$ it turns out that the following series is convergent in $L^2(\Omega, \mathcal{F}, P)$ and the sum does not depend on the chosen orthonormal system:

$$\int_0^T g_t W_t := \sum_{j=1}^\infty \int_0^T (g_t, v_j) dW_t(v_j). \tag{4}$$

We notice that each summand in the above series is a classical Itô integral with respect to a standard Brownian motion, and the resulting stochastic integral is a real-valued random variable. The independence of the terms in the series (4) leads to the isometry property

$$\mathbb{E}(|\int_0^T g_s \ dW_s|^2) = \mathbb{E}(\int_0^T \int |g_s(x)|^2 \ dx \ ds).$$

See [15] and [32] for properties of the cylindrical Wiener motion and stochastic integration.

In order to characterize the solution of the KPZ equation through a procedure of regularization, we introduce a mollified version of the cylindrical Wiener process. Let $\rho : \mathbb{R} \to \mathbb{R}$ be an infinitely differentiable function with compact support such that $\int \rho(x) \ dx = 1$. For all $n \in \mathbb{N}$ we consider the mollifier

$\delta_n(x-y) = n\rho(n(x-y))$. The mollifier cylindrical Wiener process is defined by:

$$W_t^n(x) := W_t(\delta_n(x-\cdot)). \tag{5}$$

It is a straightforward calculation to check that:

$$\mathbb{E}[W_t^n(x)W_s^n(y)] = s \wedge t C_n(x-y) \tag{6}$$

where $C_n(x) = \int \delta_n(x-u)\delta_n(-u)du$.

The quadratic variation of $W_t^n(x)$ is given by

$$< W^n(x) >_t = Cnt \tag{7}$$

where $C = \int \rho^2(-x)dx$.

We observe that for any $\varphi \in L^2(\mathbb{R})$,

$$\lim_{n\to\infty} \int W_t^n(x)\varphi(x)dx = W_t(\varphi).$$

In the case that φ has compact support the above convergence is a.e..

3 Renormalized Spaces

3.1 Renormalized distributions

We denote by $\mathcal{D}([0,T) \times \mathbb{R})$ the space of the infinitely differentiable functions with compact support in $[0,T) \times \mathbb{R}$, and $\mathcal{D}'([0,T) \times \mathbb{R})$ its dual. In order to study the KPZ equation at level of distributions, we introduce a space of renormalized distributions.

Definition 3.1 *Let*

$$\mathcal{D}_0([0,T) \times \mathbb{R}) := \{\varphi \in \mathcal{D}([0,T) \times \mathbb{R}) : \varphi = \partial_x\psi \text{ with } \psi \in \mathcal{D}([0,T) \times \mathbb{R})\}.$$

Definition 3.2 *A continuous linear functional $\phi : \mathcal{D}_0([0,T) \times \mathbb{R}) \to \mathbb{R}$ is called a renormalized distribution.*

Remark 3.1 *We observe that $\mathcal{D}_0([0,T) \times \mathbb{R})$ is equal to the set*

$$\{\varphi \in \mathcal{D}([0,T) \times \mathbb{R}) : \int \varphi(t,x) \ dx = 0\}.$$

The relationship between the theory of distributions and probability theory comes from the seminal works of K. Itô, I. Gelfand and N. Vilenkin, see [16] and [20]. They gave many significant contributions to the theory of stochastic process with values in a space of distributions. This theory has been successful in the resolution of stochastic partial differential equations driven by space-time white noise, see for instance [32]. With the purpose to give sense to the KPZ equation, we consider random variables with values in $\mathcal{D}_0'([0,T) \times \mathbb{R})$. More precisely.

Definition 3.3 *Let $\mathbf{D}$ be the space of functions $T : \Omega \to \mathcal{D}_0'([0,T) \times \mathbb{R})$ such that $< T, \varphi >$ is a random variable for all $\varphi \in \mathcal{D}_0([0,T) \times \mathbb{R})$. The elements of $\mathbf{D}$ are called random generalized functions.*

In order to define the initial condition for the KPZ equation, we need the notion of section of a distribution of S. Lojasiewicz, see [23] and [25]. We first introduce the space of test functions

$$\mathcal{D}_0(\mathbb{R}) := \{\varphi \in \mathcal{D}(\mathbb{R}) : \varphi = \partial_x \psi \ with \ \psi \in \mathcal{D}(\mathbb{R})\}$$

and its dual $\mathcal{D}_0'(\mathbb{R})$.

For convenience, we recall the notion of strict delta net, see [25] pp. 55.

Definition 3.4 *A strict delta net is a net $\{\rho_\varepsilon : \varepsilon > 0\}$ of $\mathcal{D}([0,T))$ such that it satisfies:*

1. $\lim_{\varepsilon \to 0} supp(\rho_\epsilon) = \{0\}$.

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2. *For all $\varepsilon > 0$, $\int \rho_\varepsilon(t)\, dt = 1$.*

3. $\sup_{\varepsilon > 0} \int |\rho^\varepsilon(t)|\, dt < \infty$.

Definition 3.5 *A distribution $H \in \mathcal{D}'_0([0, T] \times \mathbb{R})$ has a section $U \in \mathcal{D}'_0(\mathbb{R})$ at $t = 0$ if for all $\varphi \in \mathcal{D}_0(\mathbb{R})$ and all strict delta net $\{\rho_\varepsilon : \varepsilon > 0\}$,*

$$\lim_{\varepsilon \to 0} <H, \rho_\epsilon \varphi> \; = \; <U, \varphi>.$$

3.2 Renormalized Semimartingales

We say that a random field $\{S(t, x) : t \in [0, T], \ x \in \mathbb{R}\}$ is a spatially dependent semimartingale if for each $x \in \mathbb{R}$, $\{S(t, x) : t \in [0, T]\}$ is a semimartingale in relation to the same filtration $\{\mathcal{F}_t : t \in [0, T]\}$. If $S(t, x)$ is a C^∞-function of x and continuous in t almost everywhere, it is called a C^∞-semimartingale. The vector space of C^∞-semimartingales is denoted by **S**. See [22] for a rigorous study of spatially depend semimartingales and applications to stochastic differential equations.

In order to introduce the space of renormalized semimartingales we consider $\mathbf{S}^{\mathbb{N}_0}$ the space of sequences of C^∞-semimartingales. It is clear that $\mathbf{S}^{\mathbb{N}_0}$ has the structure of an associative, commutative differential algebra with the natural operations:

$$
\begin{aligned}
(F_n) + (G_n) &= (F_n + G_n) \\
a(F_n) &= (aG_n) \\
(F_n) \cdot (G_n) &= (F_n G_n) \\
\partial_x(F_n) &= (\partial_x F_n)
\end{aligned}
$$

where (F_n) and (G_n) are in $\mathbf{S}^{\mathbb{N}_0}$ and $a \in \mathbb{R}$.

We observe that

$\mathbf{N} = \{(G_n) \in \mathbf{S}^{\mathbb{N}_0} \ : \ \text{for each } n \in \mathbb{N}_0, G_n \text{ not dependent of the parameter } x \ a.e.\}$

is a subalgebra of $\mathbf{S}^{\mathbb{N}_0}$.

Definition 3.6 *The space of renormalized semimartingales is defined as*

$$\mathbf{G} = \mathbf{S}^{\mathbb{N}_0}/\mathbf{N}.$$

Let $(F_n) \in \mathbf{G}$. We will use $[F_n]$ to denote the equivalence class $(F_n) + \mathbf{N}$.

Example 3.1 *The mollified cylindrical Wiener process $\mathcal{W} := [W^n]$ belongs to $\mathbf{G}$.*

Proposition 3.1 *Let $[F_n] \in \mathbf{G}$. Then*

$$\partial_x[F_n] = [\partial_x F_n].$$

Proof: Let $(C_n) \in \mathbf{N}$. Then

$$\partial_x(F_n + C_n) = (\partial_x F_n + \partial_x C_n) = (\partial_x F_n).$$

$\square$

Now, we introduce the relation of association for renormalized semimartingales.

Definition 3.7 *Let $[F_n]$ and $[G_n]$ be renormalized semimartingales. We say that $[F_n]$ and $[G_n]$ are associated, (denoted by $[F_n] \approx [G_n]$) if for all $\varphi \in \mathcal{D}_0([0,T) \times \mathbb{R})$*

$$\lim_{n \to \infty} < F_n - G_n, \varphi >= 0.$$

We observe that the relation $\approx$ is well defined. In fact, if $(C_n) \in \mathbf{N}$ and $\varphi \in \mathcal{D}_0([0,T) \times \mathbb{R})$ we have that $\lim_{n\to\infty} < C_n, \varphi >= 0$. It follows immediately that $\approx$ is an equivalence relation.

Proposition 3.2 *Let $[F_n]$ and $[G_n]$ be renormalized semimartingales such that $[F_n] \approx [G_n]$. Then $\partial_x^\alpha[F_n] \approx \partial_x^\alpha[G_n]$ for all $\alpha \in \mathbb{N}$.*

Proof: By integration by parts and hypothesis,

$$\lim_{n\to\infty} < \partial_x^\alpha F_n - \partial_x^\alpha G_n, \partial_x\varphi > \;=\; \lim_{n\to\infty} < F_n - G_n, \partial_x((-1)^\alpha \partial_x^\alpha \varphi) >$$
$$=\; 0$$

for all $\varphi \in \mathcal{D}([0,T) \times \mathbb{R})$. Thus $\partial_x^\alpha[F_n] \approx \partial_x^\alpha[G_n]$. $\qquad\square$

Remark 3.2 *We would like to remark that our definition of the space of renormalized semimartingales is inspired on Colombeau's theory of nonlinear generalized functions, see for instance [8] and [13]. This theory has been successful in applications to stochastic analysis. We refer the reader to [1], [2], [10], [11], [12] and [27] for interesting works on this subject.*

In this context we have the following relation of association between renormalized semimartingales and random generalized functions.

Definition 3.8 *Let $[F_n]$ be renormalized semimartingales and T be random generalized functions. We say that $[F_n]$ is associated with T, (denoted by $[F_n] \simeq T$) if for all $\varphi \in \mathcal{D}_0([0,T) \times \mathbb{R})$,*

$$\lim_{n\to\infty} < F_n, \varphi >=< T, \; \varphi > .$$

We observe that the relation $\simeq$ is well defined. In fact, if $(C_n) \in \mathbf{N}$ and $\varphi \in \mathcal{D}_0([0,T) \times \mathbb{R})$ we have that $\lim_{n\to\infty} < C_n, \varphi >= 0$.

4 Solving the KPZ Equation

4.1 Solution of the KPZ equation in D

The distribution theory of L. Schwartz is the linear calculus of today. It is a cornerstone of modern mathematical analysis with many applications in economics, engineering and natural sciences. However, it is not possible to

define a product for arbitrary distributions with good properties, see [13], [25] and [29].

A possible approach to define a product of a pair of distributions is via regularization. That is, approximate one (or both) of the distributions to be multiplied by smooth functions, multiply this approximation by the other distribution, and pass to the limit (see for instance [3], [9], [25] and [31]). Following these ideas of regularization and passage to the corresponding limit, we introduce a new concept of solution for the KPZ equation.

Definition 4.1 *We say that $H \in \mathbf{D}$ is a renormalized-generalized solution of the equation (1) if*

1. *There exists a sequence of C^∞-semimartingales $\{H_n : n \in \mathbb{N}\}$ such that $H = \lim_{n \to \infty} H_n$ and there exists $\lim_{n \to \infty} (\partial_x H_n)^2$ in $\mathcal{D}'_0([0, T) \times \mathbb{R})$.*

2. *For all $\varphi \in \mathcal{D}_0([0, T) \times \mathbb{R})$,*

$$< H, \partial_t \varphi > + < \triangle H, \varphi > + < (\partial_x H)^2, \varphi > + \int_0^T \varphi(t, \cdot) dW_t = 0$$

   where $(\partial_x H)^2 := \lim_{n \to \infty} (\partial_x H_n)^2$.

3. *The section of H at $t = 0$ is equal to f.*

Theorem 4.1 *Let $f \in C_b^\infty(\mathbb{R})$ and Z be a solution of the stochastic heat equation (2). Then $H = \ln Z$ is a renormalized-generalized solution of (1).*

Proof: Let us denote by $H_n(t, x)$ the process $\ln Z_n(t, x)$, where Z_n is the solution of the regularized stochastic heat equation in the Itô sense

$$\begin{cases} dZ_n & = \triangle Z_n \, dt + Z_n \, dW^n, \\ Z_n(0, x) & = e^{f(x)}. \end{cases} \tag{8}$$

See [7] for an exhaustive study of the solutions of this equation.

By Itô formula and (7),

$$H_n = f + \int_0^t \triangle H_n \; ds + \int_0^t (\partial_x H_n)^2 \; ds + W_t^n - \frac{C}{2} \; nt. \tag{9}$$

Multiplying (9) by $\partial_t \varphi(t,x)$, where $\varphi \in \mathcal{D}_0([0,T) \times \mathbb{R})$, and integrating in $[0,T) \times \mathbb{R}$ we obtain that

$$< H_n, \partial_t \varphi > + < \triangle H_n, \varphi > + < (\partial_x H_n)^2, \varphi > + \int_0^T (\varphi(t,\cdot) * \delta_n) \; dW_t = 0. \tag{10}$$

We observe that Z_n converge to Z uniformly on compacts of $[0,T) \times \mathbb{R}$, see L. Bertini and N. Cancrini [7], Theorem 2.2. Thus

$$\lim_{n \to \infty} < H_n, \partial_t \varphi >=< H, \partial_t \varphi > \tag{11}$$

and

$$\lim_{n \to \infty} < \triangle H_n, \varphi >=< \triangle H, \varphi > . \tag{12}$$

We recall that $\int_0^T \varphi(t,\cdot) \; dW_t$ defines a continuous linear functional from $\mathcal{D}([0,T) \times \mathbb{R})$ to $\mathbb{R}$, see [28]. Then

$$\lim_{n \to \infty} \int_0^T (\varphi(t,\cdot) * \delta_n) \; dW_t = \int_0^T \varphi \; dW_t. \tag{13}$$

From the equation (10) and the convergences (11), (12) and (13) we have that for all $\varphi \in \mathcal{D}_0([0,T) \times \mathbb{R})$,

$$\int_0^T \int_\mathbb{R} (\partial_x H_n)^2 \; \varphi(t,x) \; dtdx$$

converges and define a linear functional in $\mathcal{D}'_0([0,T) \times \mathbb{R})$. Thus the nonlinearity

$$< (\partial_x H)^2, \varphi >:= \lim_{n \to \infty} \int_0^T \int_\mathbb{R} (\partial_x H_n)^2 \; \varphi(t,x) \; dtdx$$

is well defined.

From the continuity of $Z(t,x)$, see for instance [7], we observe that

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$$\lim_{\varepsilon \to 0} \int \int_0^T ln \ Z(t,x) \ \rho_\varepsilon(t) \ dt \ \varphi(x) \ dx = \int f(x) \ \varphi(x) \ dx,$$

for all $\varphi \in \mathcal{D}_0(\mathbb{R})$ and for all strict delta net. Thus we conclude that H is a renormalized-generalized solution for the problem (1).

Remark 4.1 *The solution $Z(t,x)$ of the stochastic heat equation (2) is understood in a mild sense (see for example [7], [15] and [30]).*

4.2   Solution of the KPZ equation in G

Definition 4.2 *We say that $F \in \mathbf{G}$ is a renormalized solution of the KPZ-equation (1) if*

$$F = f + \int_0^t \triangle F \ ds + \int_0^t (\partial_x F)^2 \ ds + \ \mathcal{W}$$

where the equality holds in $\mathbf{G}$.

Theorem 4.2 *For every $f \in C_b^\infty(\mathbb{R})$ there exist a renormalized solution F in $\mathbf{G}$ for the KPZ-equation (1).*

Proof: Let Z_n be the solution of the regularized stochastic heat equation (8). We denote $ln \ Z_n(t,x)$ by $F_n(t,x)$ and $[F_n]$ by F. We claim that F is a renormalized solution of the KPZ equation. In fact, by the Itô formula we have that

$$F_n = f + \int_0^t \triangle F_n \ ds + \int_0^t (\partial_x F_n)^2 \ ds + \ W_t^n - \frac{C}{2} \ nt. \qquad (14)$$

Taking equivalence class, we obtain

$$F = f + \ [\int_0^t \triangle F_n \ ds] + \ [\int_0^t (\partial_x F_n)^2 \ ds] + \ \mathcal{W}.$$

We need to check that

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a) $[\int_0^t (\partial_x F_n)^2 \ ds] = \int_0^t (\partial_x [F_n])^2 \ ds$

b) $[\int_0^t \triangle F_n \ ds] = \int_0^t \triangle [F_n] \ ds.$

a) We observe that

$$\int_0^t (\partial_x F_n)^2 \ ds + D_n - \int_0^t (\partial_x (F_n + C_n))^2 \ ds$$

is equal to

$$\int_0^t (\partial_x F_n)^2 \ ds + D_n - \int_0^t (\partial_x F_n)^2 \ ds = D_n.$$

b) The proof is analogous to a).

Remark 4.2 *We observe from the proof of the Theorem 4.2 that* $F = [\ln Z_n]$ *is a renormalized solution of the KPZ equation. Moreover, we have that for all* $\varphi \in \mathcal{D}_0([0,T) \times \mathbb{R})$ *it holds*

$$\lim_{n \to \infty} < \ln Z_n, \varphi > = < \ln Z, \ \varphi > .$$

Hence we get $[\ln Z_n] \simeq \ln Z$ *in the sense of Definition 3.8.*

5 End Comments and Future Work

5.1 Others Approaches

Very recently, two other ways to make sense of the Cole-Hopf solution were given.

S. Assing, in [4], introduces a weak solution, in a probabilistic sense, for the conservative version of the equation (1). The idea is to approximate the Cole-Hopf solution by the density fluctuations in weakly asymmetric exclusion. In [17], P. Gonçalves and M. Jara considered a similar type of solution.

M. Hairer, in [18], introduced a notion of pathwise solution for the KPZ equation via the rough path theory. However, the results of [18] are for KPZ on $[0, 2\pi]$ with periodic boundary conditions, and extending them to $\mathbb{R}$ remains an open problem.

5.2   Solution of the KPZ Equation for $d > 1$.

The stochastic heat equation in $\mathbb{R}^d$, is given by

$$
\begin{cases}
\partial_t Z & = \; \triangle Z + Z \, W \\
Z(0, x) & = \; e^{f(x)}
\end{cases}
\tag{15}
$$

where $W(t, x)$ is the space-time white noise in $\mathbb{R}^d$.

We observe that the Cole-Hopf solution does not make sense because the solution of the stochastic heat equation is not a standard stochastic process. It is realized as a generalized stochastic process in the space of stochastic Hida distribution (see for example [19]).

Thus, we expect solutions of KPZ equation only in the sense of Colombeau's generalized functions. Stochastic processes whose paths are generalized functions are considered in [1] and [2], and are used for solving some classes of nonlinear stochastic equations. For a new approach see [10], [11] and [12].

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